Journal of Engineering Mathematics, Vol. 11, No. 2, April 1977
Noordhoff International Publishing–Leyden
Printed in The Netherlands

The disturbance produced by an oscillatory pressure distribution in uniform translation on the surface of a liquid

A. H. MAGNUSON

Naval Ship Research and Development Center, Bethesda, Md. 20084, U.S.A.

(Received September 17, 1975 and in revised form February 25, 1976)

SUMMARY

Expressions are derived for the two-dimensional surface elevation resulting from an oscillatory translating surface pressure distribution. The surface elevation is given as the sum of four terms, each of which is associated with an improper integral having a simple pole singularity. Results are presented for the delta function and the uniform spatial pressure distribution.

The mean work done on the fluid per unit time by the delta function pressure distribution is given. Numerical results are presented for the surface elevation resulting from the uniform pressure distribution.

1. Introduction

The current development of air cushion supported marine vehicles has kindled a renewed interest in the analysis of water waves produced by surface pressure distributions. Initially the primary application was for the prediction of wavemaking drag produced by translating pressure distributions having various planform shapes. More recently, unsteady problems have been investigated in order to obtain insight into various aspects of the dynamic performance of air cushion supported vehicles [1], [2].

Stoker, [3] Kaplan, [4] and Wu [5] treated the two-dimensional problem for a harmonic, uniformly translating delta function pressure distribution. Kaplan obtained expressions for the surface elevation in the far field while Wu obtained asymptotic results for the velocity potential and surface elevation in both the near and far field. Stoker discussed the qualitative behavior of the solution, and obtained results for the near and far field for the zero speed case.

Debnath and Rosenblatt [6] treated the two-dimensional finite depth problem using generalized function theory to obtain an asymptotic solution. The same technique has been applied recently by Pramanik [7] to the two-layer fluid.

Lighthill [8] analyzed the qualitative nature of the three-dimsional wave pattern produced by an unsteady translating pressure distribution. More recently, Tayler and Van den Driessche [9] used ray theory to obtain qualitative results for the three-dimensional wave pattern produced by a periodic translating submerged source. The problem treated in this paper is closely related to the translating submerged source of pulsating strength. A useful survey and discussion of the literature on the pulsating source has been compiled by Wehausen and Laitone [10].

This paper presents, for the first time, an entirely analytical result for the two-dimensional surface elevation that is valid for the entire field. Results are given for both a delta function and a uniform pressure distribution. It was possible to obtain an analytical solution to the problem by directly evaluating improper integrals arising from a Fourier spatial transform. The integrals were evaluated by a careful examination of the singularities, algebraic manipulation and proper choice of contour of integration. The results are given in terms of well-known transcendental functions. The results will be extended to three dimensions in the future, but a detailed analysis of the two-dimensional dispersion was considered necessary to clarify the fundamental nature of the wave field. The methodology used is equally applicable to the three-dimensional problem.

The two-dimensional problem is treated here for an irrotational, incompressible inviscid fluid of infinite depth using linearized potential theory. The formulation of the problem generally follows Doctors' three dimensional analysis [2] and also parallels Wu's [5] two-dimensional analysis. The major departure from Doctors and Wu comes in the evaluation of the integral forms by contour integration and the application of Cauchy's theorem.

2. Derivation of velocity potential and surface elevation

The coordinate system and notation are shown in Fig. 1. The surface pressure distribution $p(x, t)$ is translating to the right with a speed c relative to the fixed coordinate system. The surface elevation is denoted as $z(x, t)$. The fluid is considered to be irrotational and incompressible so that a velocity potential $\Phi(x, z, t)$ exists. The potential obeys the following relation:

$$\Delta_2 \Phi(x, z, t) = 0, \tag{1}$$

where Δ_2 is the two-dimensional Laplacian operator. The fluid velocity is the gradient of the potential, or

$$u = \Phi_x, \quad w = \Phi_z, \tag{2}$$

where u is the x component and w is the z component of the velocity, and the subscripts x and z represent partial differentation in each respective direction.

The kinematic free surface boundary condition is written in linearized form as follows:

$$(\Phi_z)_{z=0} + cZ_x - Z_t = 0. \tag{3}$$

The linearized dynamic boundary condition is:

$$(\Phi_t - c\Phi_x + \mu\Phi)_{z=0} = -\left(\frac{p}{\rho} + gZ\right), \tag{4}$$

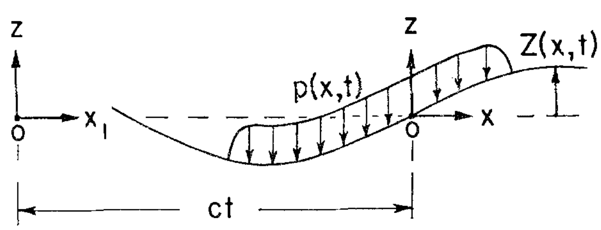


Figure 1. Coordinate system and notation for the translating oscillatory surface pressure distribution.

where μ is the Rayleigh viscosity. The temporary introduction of Rayleigh viscosity proves to be useful in the interpretation of improper integrals derived later in the paper.* The combined free surface boundary condition may be written as

$$[\Phi_{tt} - 2c\Phi_{xt} + c^2\Phi_{xx} + g\Phi_z + \mu(\Phi_t - c\Phi_x)]_{z=0} = \frac{-1}{\rho}(p_t - cp_x). \tag{5}$$

One may also write the following condition for the fluid of infinite depth:

$$\Phi_z = 0, \quad z \to -\infty. \tag{6}$$

Since the case of harmonic time dependence is being treated, one may write for the pressure

$$p(x, t) = p(x)e^{-i\sigma t}, \tag{7}$$

where σ is the radian frequency.

It is convenient to solve for the response due to a delta function pressure distribution $p(x) = \delta(x)$. The resulting surface elevation will be denoted $\zeta(x, t)$ and the velocity potential $\phi(x, z, t)$. The response to an arbitrary spatial pressure distribution $p(x)$ may be obtained from ζ and ϕ using superposition integrals as follows:

$$\Phi(x, z, t) = \int_{-\infty}^{\infty} p(\xi)\,\phi(x - \xi, z, t)\,d\xi$$

and (8)

$$Z(x, t) = \int_{-\infty}^{\infty} p(\xi)\,\zeta(x - \xi, t)\,d\xi.$$

One may formally express the velocity potential and surface elevation in a Fourier integral form:

$$\phi(x, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi'(k, z, t)e^{ikx}dk \tag{9}$$

and

$$\zeta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta'(k, t)e^{ikx}dk, \tag{10}$$

where ϕ' and ζ' are the transformed potential and surface elevation. From (1) and (6), one sees that the transformed potential may be written

$$\phi' = A(k, t)e^{|k|z}. \tag{11}$$

* The ficticious Rayleigh viscosity is frequently used as a mathematical artifice to shift pole singularities off the real axis. This enables one to properly interpret the integration path after the ficticious viscosity is removed. (See Wehausen and Laitone [10], p. 479). The problem could have been formulated using the Navier–Stokes equations and modified boundary conditions similar to Miles' treatment of the Cauchy–Poisson problem [11]. A treatment of this sort would be more satisfactory from a physical standpoint. Either way, one obtains the same result after suppressing the viscosity. The introduction of a weak internal damping mechanism is not uncommon in other areas of classical physics. For example, the author in Reference [12] has applied Voigt viscoelasticity to the problem of acoustic reflection from a solid halfspace. The dissipative mechanism facilitated the interpretation of a spatial Fourier integral form.

As one is concerned only with the steady state response, the time dependence of the velocity potential and surface elevation will be of the same form as the applied pressure given in equation (7). In this case the time differentiation indicated in the boundary conditions reduces to multiplication by $-i\sigma$.

The transformed surface elevation may then be expressed as follows from equations (3) and (11):

$$\zeta'(k,t) = \frac{i|k|\,A(k,t)}{\sigma + kc}. \tag{12}$$

One solves for the potential function $A(k, t)$ by substituting (11) and (7) into the transformed form of the combined free surface boundary condition (5), giving as a result

$$A(k,t) = \frac{\mathrm{e}^{-i\sigma t}}{\rho}\,\frac{i(kc+\sigma)}{\{(|k|g - \mu^2/4 - (\sigma + kc + i\mu/2)^2\}}. \tag{13}$$

From (12) and (13), the transformed surface elevation is:

$$\zeta'(k,t) = \frac{\mathrm{e}^{-i\sigma t}}{\rho}\,\frac{-|k|}{\{|k|g - \mu^2/4 - (\sigma + kc + i\mu/2)^2\}}. \tag{14}$$

To eliminate the absolute signs in equation (14) one may break the integral form (10) into two regions, one along the negative real axis and the other along the positive real axis. After some manipulation the following result is obtained:

$$\zeta(x,t) = \frac{\mathrm{e}^{-i\sigma t}}{2\pi\rho}\left\{\int_0^\infty \frac{\mathrm{e}^{ikx}\,k\,dk}{D_1(k,\sigma,c)} + \int_0^\infty \frac{\mathrm{e}^{-ikx}\,k\,dk}{D_2(k,\sigma,c)}\right\}, \tag{15}$$

where

$$D_1 = (\sigma + kc)^2 + i\mu(\sigma + kc) - kg$$

and

$$D_2 = (\sigma - kc)^2 + i\mu(\sigma - kc) - kg.$$

3. Evaluation of integrals by contour integration

To evaluate the integrals in equation (15) one must first determine the nature of the singularities of the integrands. No branch point singularities are evident, but the two denominators D_1 and D_2 each have two distinct zeros corresponding to simple pole singularities. The second denominator may be written as follows:

$$D_2(k,\sigma,c) = c^2(k - K_1)(k - K_2), \tag{16}$$

where

$$K_1 = K_0(2 + \alpha - 2\sqrt{1+\alpha}), \quad K_2 = K_0(2 + \alpha + 2\sqrt{1+\alpha}),$$

$$K_0 = g/(4c^2), \quad \alpha = \alpha_0(1 + i\varepsilon), \quad \alpha_0 = \frac{4c\sigma}{g}, \quad \varepsilon = \mu/(2\sigma).$$

The damping is taken to be small, so $\varepsilon \ll 1$.

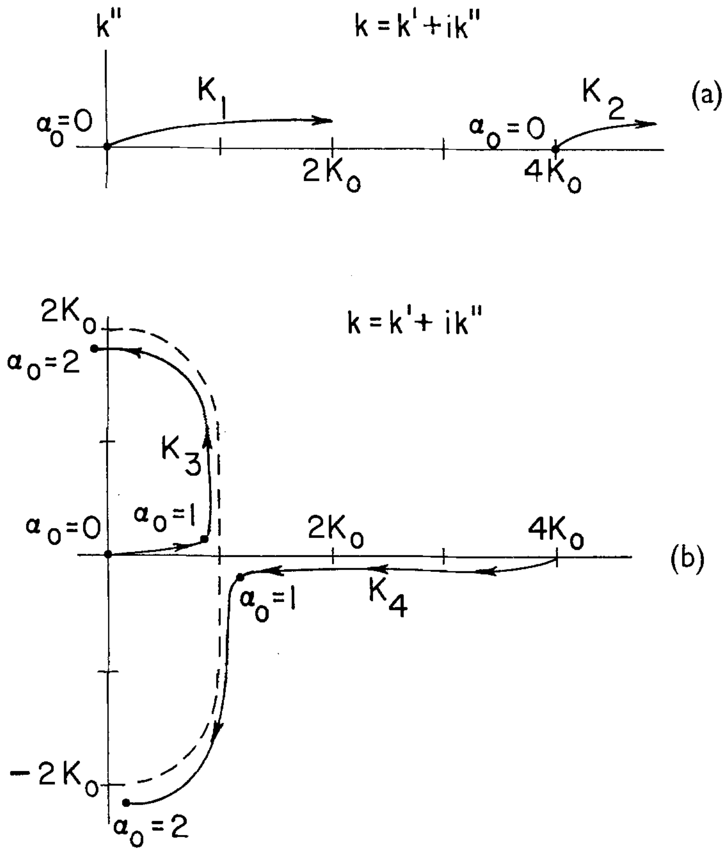


Figure 2. Migration of poles in complex wavenumber plane as frequency parameter α_0 increases (a) Migration of K_1 and K_2; (b) Migration of K_3 and K_4.

The migrations of K_1 and K_2 as the frequency parameter α_0 increases are shown in Fig. 2a. One sees that both poles remain near the real axis for all values of α_0.

One may write for the first denominator:

$$D_1(k, \sigma, c) = c^2(k - K_3)(k - K_4), \tag{17}$$

where

$$K_3 = K_0(2 - \alpha - 2\sqrt{1 - \alpha}), \quad K_4 = K_0(2 - \alpha + 2\sqrt{1 - \alpha}).$$

The migration of these poles as α_0 increases is shown in Fig. 2b. The poles remain near the real axis until α_0 approaches unity, at which point both poles begin to move away from the real axis.

The expression for the wave elevation may be simplified by substituting (16) and (17) into (15) and applying a partial fraction expansion to each term. The resulting expression for the surface elevation is

$$\zeta(x, t) = \frac{e^{-i\sigma t}}{2\pi\rho g}\left\{-\frac{K_1}{\sqrt{1+\alpha}} I_1(x, K_1) + \frac{K_2}{\sqrt{1+\alpha}} I_2(x, K_2) - \frac{K_3}{\sqrt{1-\alpha}} I_3(x, K_3) + \frac{K_4}{\sqrt{1-\alpha}} I_4(x, K_4)\right\}, \tag{18}$$

where

$$I_1(x, K_1) = \int_0^\infty \frac{e^{-ikx}}{k - K_1}\, dk, \quad I_2(x, K_2) = \int_0^\infty \frac{e^{-ikx}}{k - K_2}\, dk,$$

$$I_3(x, K_3) = \int_0^\infty \frac{e^{ikx}}{k - K_3}\, dk, \quad I_4(x, K_4) = \int_0^\infty \frac{e^{ikx}}{k - K_4}\, dk.$$

The problem now reduces to evaluation of the improper integrals I_1, I_2, I_3, and I_4 given in (18). At this point the artificial internal dissipation may be eliminated by setting the Stokes viscosity coefficient μ to zero. The dissipation was introduced to determine in which quadrant the poles K_1, K_2, K_3, and K_4 lie. When the Stokes viscosity μ is set to zero the nondimensional frequency parameter α in equation (16) becomes real:

$$\alpha = \alpha_0 = \frac{4\sigma c}{g}. \tag{19}$$

The poles K_3 and K_4 now lie on the real axis for $\alpha \leqq 1$ and K_1 and K_2 are real for all α. For $\alpha > 1$, the complex poles K_3 and K_4 may be expressed in exponential form as follows:

$$K_3 = \alpha K_0 e^{i\psi}, \quad K_4 = \alpha K_0 e^{-i\psi}, \quad (\alpha > 1), \tag{20}$$

where

$$\psi = \tan^{-1}\left\{\frac{2\sqrt{\alpha - 1}}{2 - \alpha}\right\}.$$

The paths of integration for the improper integrals in (18) must be reinterpreted when the artifical dissipation is eliminated. The K_1 and K_2 poles for all values of α and the K_3 and K_4 poles for $\alpha \leqq 1$ now lie on the real axis, so the paths must be indented. Each path is idented so that the pole lies on the same side of the integration path as it did when dissipation was present. The expressions for the integrals may be written as follows for the non-dissipative medium:

$$I_1(x, K_1) = \int_{\Gamma_1} \frac{e^{-ikx}}{k - K_1}\, dk, \quad I_2(x, K_2) = \int_{\Gamma_2} \frac{e^{-ikx}}{k - K_2}\, dk,$$

$$I_3(x, K_3) = \int_{\Gamma_3} \frac{e^{ikx}}{k - K_3}\, dk, \quad I_4(x, K_4) = \int_{\Gamma_4} \frac{e^{ikx}}{k - K_4}\, dk, \tag{21}$$

where the paths Γ_1, Γ_2, Γ_3, and Γ_4 are now in the complex k-plane as indicated in Fig. 3.

The following symmetries exist between the integrals in equation (21). These follow from the paths shown in Fig. 3 and equation (21).

$$I_2(x, K) = I_1(x, K), \quad I_3(x, K) = I_1(-x, K), \quad I_4(x, K) = I_3^*(-x, K), \tag{22}$$

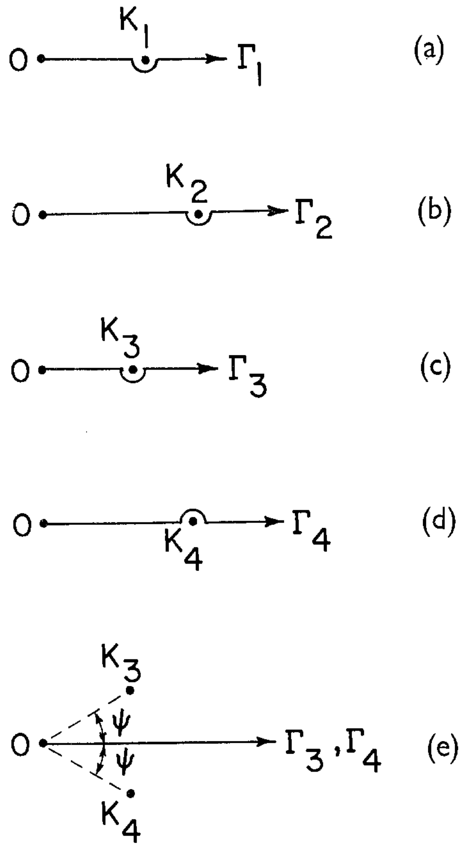


Figure 3. Integration paths in complex wavenumber plane for dissipationless liquid
(a) Path for I_1; (b) Path for I_2; (c) Path for I_3, $\alpha \leqq 1$; (d) Path for I_4, $\alpha \leqq 1$; (e) Path for I_3 and I_4 for $\alpha > 1$.

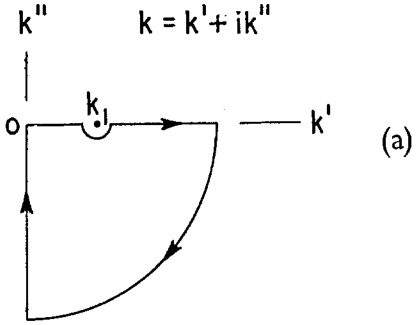
where the asterisk denotes a complex conjugate and K is real. The symmetry relations reduce the number of integrations from eight to two.

One starts by integrating I_1, as given in equation (21). The behavior of the exponential term is exploited. One notes that for positive x the exponential of the integrand for I_1 has a negative real part in the lower half of the complex k plane. For $x > 0$, I_1 is evaluated by closing a contour in the fourth quadrant as shown in Fig. 4a. The pole K_1 is excluded from the contour because of the indentation. After applying Cauchy's integral theorem around the closed contour, one has

$$I_1^+ + \int_{-i\infty}^{0} \frac{e^{-ikx}dk}{k - K_1} = 0, \tag{23}$$

where the superscript plus sign denotes the solution on the positive x axis. After some manipulation, the integral I_1^+ may be expressed in terms of auxiliary exponential integrals [13] as follows:

$$I_1^+(x, K_1) = g(K_1 x) + if(K_1 x). \tag{24}$$



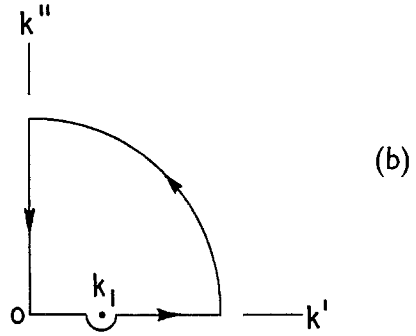


Figure 4. Integration contours in complex wavenumber plane for I_1
(a) Contour for I_1^+; (b) Contour for I_1^-.

One evaluates I_1 for $x < 0$ (denoted I_1^-) in a similar fashion. The contour is closed in the first quadrant as shown in Fig. 4b. Noting that the pole is now inside the contour, one may apply Cauchy's residue theorem, giving

$$I_1^- + \int_{i\infty}^{0} \frac{e^{-ikx}\,dk}{k - K_1} = 2\pi i e^{-iK_1 x}. \tag{25}$$

After similar manipulation, the expression for I_1^- reduces to

$$I_1^-(x, K_1) = g(-K_1 x) + i[2\pi e^{-iK_1 x} - f(-K_1 x)], \qquad x < 0. \tag{26}$$

Because of symmetry (22), the expressions for I_2^+ and I_2^- may be obtained from (24) and (26) by substituting K_2 for K_1.

The second symmetry relation (22) is exploited to obtain the following expressions for I_3^+ and I_3^- for $\alpha \leqq 1$:

$$I_3^+(x, K_3) = g(K_3 x) - i[f(K_3 x) - 2\pi e^{iK_3 x}] \tag{27}$$

and

$$I_3^-(x, K_3) = g(-K_3 x) + if(-K_3 x). \tag{28}$$

Finally, the third symmetry relation is used to evaluate I_4^+ and I_4^- for $\alpha \leqq 1$:

$$I_4^+(x, K_4) = g(K_4x) - if(K_4x) \tag{29}$$

and

$$I_4^-(x, K_4) = g(-K_4x) + i[f(-K_4x) - 2\pi e^{iK_4x}]. \tag{30}$$

To evaluate I_3 and I_4 for $\alpha > 1$ the locations of the complex poles K_3 and K_4 must be taken into account. From Figs. 2b and 3e and equation (20) one sees that K_3 lies in the upper half plane and K_4 in the lower half plane. Both poles lie in the right half plane for $\alpha < 2$ and in the left half plane for $\alpha > 2$.

For $\alpha < 2$ the expressions for I_3 and I_4 given in (27), (28), (29) and (30) apply because the poles still lie in the same quadrants as they did for $\alpha \leqq 1$. However, the arguments of the auxiliary exponential integrals become complex and the behavior of the residue terms changes because the poles K_3 and K_4 are complex. The residue terms in equations (27) and (30) for I_3^+ and I_4^-, instead of representing unattenuated surface waves, now have exponential attenuation as one moves away from the disturbance.

For $\alpha > 2$ the K_3 and K_4 poles both lie in the left half plane. No residue terms occur in this case, as neither pole lies inside the integration contour. Therefore, both I_3 and I_4 consist solely of auxiliary exponential integral terms with complex arguments. Equations (27) and (30) still apply if the residue terms are dropped in the expressions for I_3^+ and I_4^-.

4. Mean work rate and radiation of energy into the far field

The rate at which energy is carried away from the pressure disturbance by the free wave system is of importance because of its association with the work done by the pressure on the fluid in the near field. In the far field the disturbance consists of four surface waves for $0 < \alpha \leqq 1$ and two for $\alpha > 1$. The energy efflux may be written following Lamb [14], as:

$$\overline{W}_{\mathrm{OUT}} = \sum_{n=1}^{4} c_{g_n} E_n, \tag{31}$$

where $\overline{W}_{\mathrm{OUT}}$ is the mean energy efflux through the boundary of a control volume moving with the disturbance, c_{g_n} is the group velocity of the nth wave in the moving coordinate system and E_n is the mean energy per unit surface area. In equation (31) the relative group velocity is taken as positive when directed away from the disturbance. Calculating the group velocities from the wavenumber expressions (16) and (17) gives the following:

$$\begin{aligned} c_{G1} &= \frac{-c\sqrt{1+\alpha}}{\sqrt{1+\alpha}-1}, \quad c_{G2} = \frac{-c\sqrt{1+\alpha}}{1+\sqrt{1+\alpha}}, \\ c_{G3} &= \frac{c\sqrt{1-\alpha}}{1-\sqrt{1-\alpha}}, \quad c_{G4} = \frac{-c\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}, \quad (\alpha \leqq 1). \end{aligned} \tag{32}$$

The mean energy for each wave may be expressed as follows:

$$E_n = \tfrac{1}{2}\rho g A_n^2, \tag{33}$$

where the amplitudes of the waves A_1, A_2, A_3 and A_4 are taken from (26), (27), (30) and (18). Substituting the group velocities (32) and the energies (33) into the work rate expression (31) gives the following:

$$\overline{W}_{\text{OUT}} = \frac{g}{2\rho c^3} f(\alpha), \tag{34}$$

where $f(\alpha) = f_1(\alpha) + f_2(\alpha)$ and

$$f_1(\alpha) = \frac{1}{2}\left(1 + \frac{\alpha}{4}\right),$$

$$f_2(\alpha) = \begin{cases} \dfrac{(\frac{1}{2} - \frac{3}{8}\alpha)}{\sqrt{1-\alpha}}, & \alpha \leqq 1 \\ 0, & \alpha > 1. \end{cases}$$

The normalized work rate function $f(\alpha)$ is shown in Fig. 5. One sees the familiar resonance at $\alpha = 1$. The resonance occurs because the work performed by the pressure distribution on the fluid creates energy that cannot propagate away from the disturbance. This follows from equation (32) where one sees that the relative group velocities of the third and fourth waves are zero for $\alpha = 1$. The expressions for the group velocities (32) and the mean work rate (34) are consistent with Wu's [5] results.

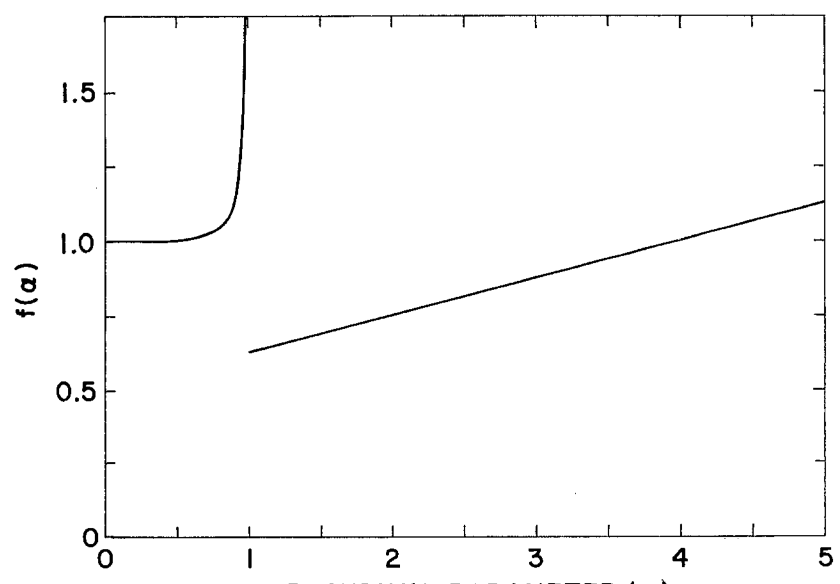


Figure 5. Normalized mean work rate function $f(\alpha)$ as a function of the frequency parameter α.

5. The surface elevation for the uniform pressure distribution

The surface elevation $Z(x, t)$ due to a uniform pressure distribution was computed from $\zeta(x, t)$ using the superposition integral (8). The surface elevation caused by the delta function distribution $\zeta(x, t)$ was obtained from (18), (24), (26), (22) and (27)–(30). One may write the uniform pressure distribution as

$$p(x) = \begin{cases} p_0, & |x| \leqq l/2 \\ 0, & |x| > l/2. \end{cases} \tag{35}$$

The results are expressed in nondimensional form as follows:

$$z' = \frac{Z}{p_0/\rho g} = (a + ib)\mathrm{e}^{-i\sigma t}, \tag{36}$$

where z' is the normalized wave elevation, a is the component in phase with the pressure and b is the out of phase component. Expressions for the in and out of phase components for $\alpha \leqq 1$ are given in the Appendix. The two components of the wave elevation are functions of the normalized frequency α and the Froude number

$$F = c/(gl)^{\frac{1}{2}}.$$

Numerical results for $F = 0.7$ are shown in Figs. 6, 7, and 8 for three frequencies: $\alpha = 0$, 0.5 and 0.95. In each figure the normalized in phase (a) and out of phase component (b) are plotted as a function of the nondimensional distance $x' = x/l$. Fig. 5 shows the in phase component of the wave elevation for the zero frequency case. (The out of phase component is zero.) The near field disturbance resembles the wake produced by a planing surface. The standing wave in the far field downstream is evident. Figs. 6 and 7 show the in phase and out of phase components of the wave elevation for $\alpha = 0.5$ and 0.95, respectively. One can see interference between the various waves in the far field downstream for both frequencies. In addition, a long wave appears upstream for $\alpha = 0.95$, but is not apparent for $\alpha = 0.5$.

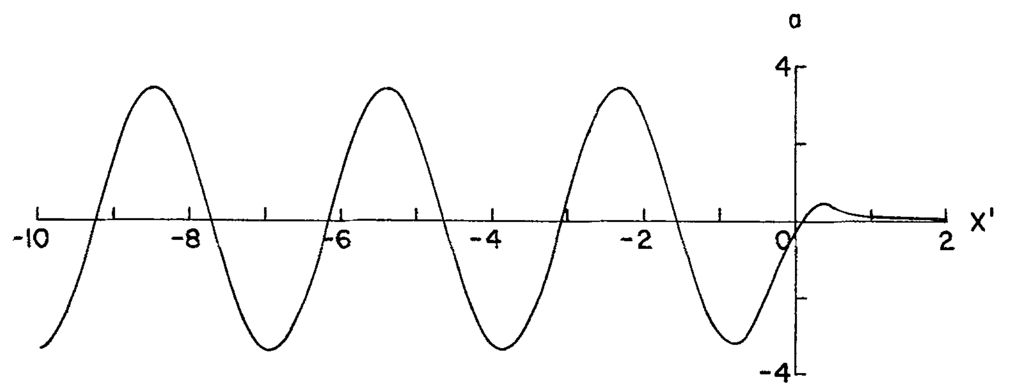


Figure 6. Normalized wave elevation for $F = 0.7$ and $\alpha = 0$.

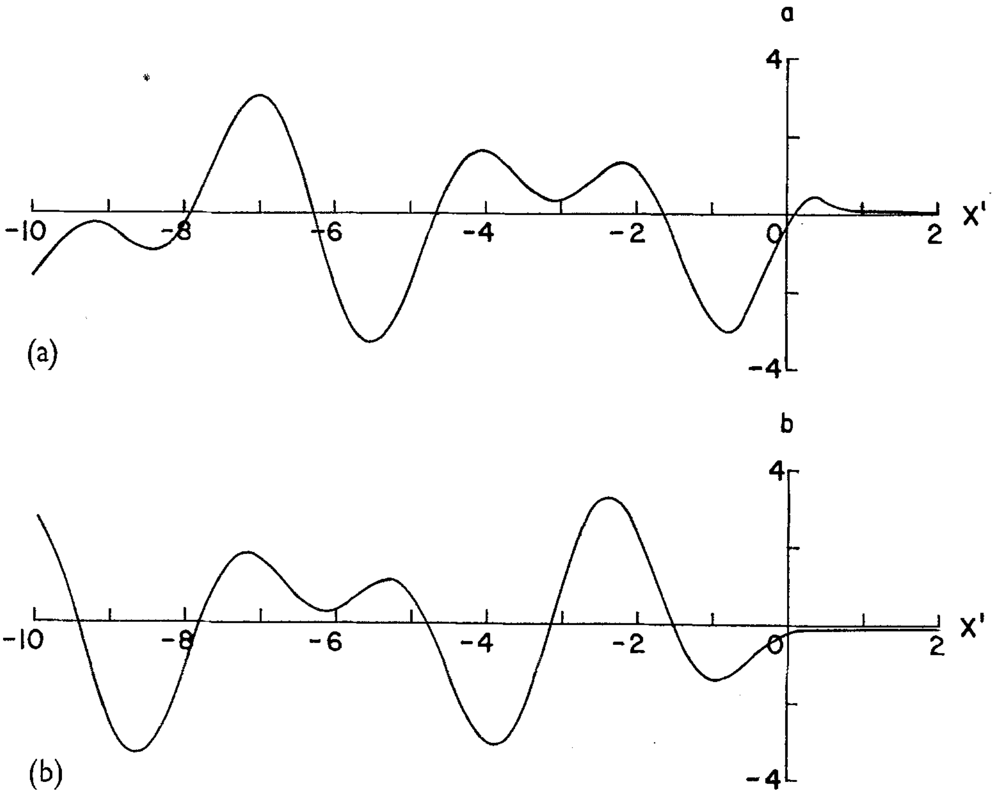


Figure 7. Normalized wave elevation for $F = 0.7$ and $\alpha = 0.5$
(a) In phase component; (b) Out of phase component.

To clarify the behavior of the waves in the far field, Table 1 has been prepared. The ratio of wavelength to pressure distribution length (λ/l) has been calculated for each wave at all three frequencies using the wavenumber expressions (16) and (17). One sees from the table that the second and fourth wave have the same wavelength and form a standing wave for $\alpha = 0$, because one wave travels to the right and the other to the left. The first and third waves have infinite wavelength, but their amplitudes are zero.

For nonzero frequencies, the length of the second wave is shorter than the fourth. This causes the interference pattern in the downstream wake. The third wave occurs upstream. Its amplitude is too small to appear in Fig. 6, but it is evident in Fig. 7. The wave is so long at $\alpha = 0.95$ that only about half a cycle of the wave appears in the figure.

TABLE 1

Wavelengths in far field produced by uniform pressure distribution

α	λ_1/l	λ_2/l	λ_3/l	λ_4/l
0	∞	3.08	∞	3.08
0.5	244	2.49	144	4.23
0.95	78.3	2.14	20.4	4.93

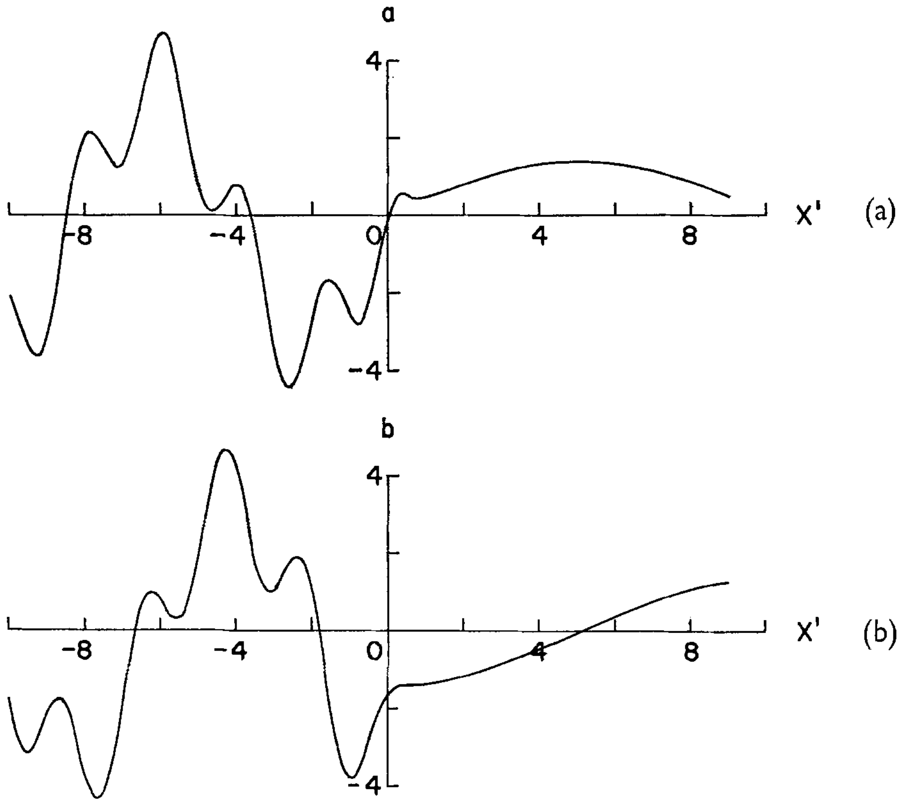


Figure 8. Normalized wave elevation for $F = 0.7$ and $\alpha = 0.95$
(a) In phase component; (b) Out of phase component.

6. Recapitulation of results

The steady state surface elevation is expressed in (18) as the sum of four integral terms, each having a simple pole singularity in the complex wavenumber plane. The location of the poles is shown in Fig. 2. When the Stokes viscosity is suppressed all the poles lie on the positive real axis for low frequencies ($\alpha \leqq 1$). At higher frequencies ($\alpha > 1$) only the first two lie on the real axis. The integration paths are indented as shown in Fig. 3 so that the poles lie on the same side of the path as they did with Stokes viscosity. Each integral term is then interpreted as a path integral in the complex wavenumber space as indicated in equation (21). To reduce the number of integrations, three symmetry relations that follow from the integral forms and the integration paths are introduced.

The first integral form is evaluated by selecting appropriate closed contours for the positive and negative x-axis, as shown in Fig. 4, and applying Cauchy's residue theorem. The integral is expressed in discontinuous form in equations (24) and (26). The integral consists partially of auxiliary exponential integrals which are known tabulated functions.

In addition a residue term appears on the negative x-axis or the downstream side. This term is an exponential and it represents the familiar undamped surface wave. The other three integrals are written from the first result using the symmetry relations.

The surface elevation is seen to consist of exponential integral terms contributing to the near field and exponentials appearing on only one side of the disturbance. The exponential terms are the free waves and are associated with the residues of each pole. For low frequencies ($\alpha \leqq 1$) three waves appear downstream and one upstream, as has been noted by previous investigators. [3, 4, 5]

At the higher frequencies two of the poles leave the real axis. The exponential integral terms in the solution remain the same, except that their arguments become complex instead of real. The residue terms persist until $\alpha = 2$, at which point the poles leave the integration contour. For $1 < \alpha < 2$ the residue terms decay exponentially with distance and, therefore, do not contribute to the far field solution.

The mean rate at which energy propagates away from the disturbance was computed. The result, which agrees with Wu [5], is given in equation (34), and is shown in Fig. 5.

The surface elevation was calculated for a uniform pressure distribution. This was done to eliminate the logarithmic singularities that appear in the solution for the delta function distribution. Expressions for the normalized in phase and out of phase components of the surface elevation are given in the appendix for the low frequency case. Numerical results are presented in Figs. 6, 7, and 8 for a Froude number of 0.7 and $\alpha = 0$, 0.5, and 0.95, respectively. The zero frequency result ($\alpha = 0$) shows that the water surface deforms like that of a planing surface in the near field, while standing waves are apparent in the far field. For the nonzero frequencies ($\alpha = 0.5$ and 0.95) interference occurs in the waves downstream. In addition, a long wave occurs upstream for $\alpha = 0.95$.

The lengths of the various waves were calculated for the three frequencies. The results are shown in the table. The zero frequency standing wave is shown to consist of two downstream waves, each having the same wavelength and traveling in opposite directions. The interference pattern in the downstream wave pattern for nonzero frequency is caused primarily by the same two waves, whose wavelengths now differ. A long wave appears upstream as one approaches the critical frequency.

Acknowledgments

The author wishes to thank M. D. Ochi and Z. G. Wachnick for their support. Helpful suggestions from Prof. J. N. Newman are gratefully acknowledged. The computer programming for the surface elevation calculation was performed by R. T. Waters. The work was sponsored by the Systems Development Department of the David W. Taylor Naval Ship Research and Development Center.

REFERENCES

[1] L. J. Doctors and S. D. Sharma, The wave resistance of an air-cushion vehicle in steady and accelerated motion, *Journal of Ship Research*, Vol. 16 (1972) 248–260.

[2] L. J. Doctors, The hydrodynamic influence on the non-linear motion of an ACV over waves, *Proceedings of the Tenth Symposium on Naval Hydrodynamics*, Office of Naval Research, Washington, 1974 (Preprint).

[3] J. J. Stoker, *Water Waves*, Interscience Publishers, Inc., New York 1957.

[4] P. Kaplan, The waves generated by the forward motion of oscillatory pressure distributions, *Proceedings of Fifth Midwestern Conference on Fluid Mechanics*, University of Michigan (1957) 316–329.

[5] T. Y. Wu, *Water Waves Generated by the Translatory and Oscillatory Surface Disturbance*, Engineering Division, California Institute of Technology Report No. 85–3, July 1957.
[6] L. Debnath and S. Rosenblatt, The ultimate approach to the steady state in the generation of waves on a running stream, *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. XXII (1969) 221–233.
[7] A. K. Pramanik, Waves due to a moving oscillatory surface pressure in a statified fluid, *Journal of Applied Mechanics*, Vol. 41 (1974) 571–574.
[8] M. J. Lighthill, On waves generated in dispersive systems by travelling forcing effects, with applications to the dynamics of rotating fluids, *Journal of Fluid Mechanics*, Vol. 27 (1967) 725–752.
[9] A. B. Tayler and P. van den Driessche, Small amplitude surface waves due to a moving source, *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. XXVII (1974) 317–345.
[10] J. V. Wehausen and E. V. Laitone, Surface waves, *Encyclopedia of Physics*, Vol. IX, Fluid Dynamics III, Springer-Verlag, Berlin (1960) 479–495.
[11] J. W. Miles, The Cauchy–Poisson problem for a viscous liquid, *Journal of Fluid Mechanics*, Vol. 34 (1968) 359–370.
[12] A. H. Magnuson, Acoustic response in a liquid overlying a homogeneous viscoelastic half-space, *Journal of the Acoustical Society of America*, Vol. 57 (May 1975) 1017–1024.
[13] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55, June 1964, Chapter V.
[14] H. Lamb, *Hydrodynamics*, Sixth Edition, Dover Publications, New York (1932) 382–384.

Appendix: Expressions for in phase and out of phase components of surface elevation for uniform pressure distribution

The normalized in phase component of the surface elevation is given as $a(F, \alpha)$ and the out of phase component as $b(F, \alpha)$. Each component consists of four parts corresponding to the four poles K_1, K_2, K_3 and K_4. One may write for a and b:

$$a = \sum_{i=1}^{4} a_i$$

and

$$b = \sum_{i=1}^{4} b_i.$$

As a result of the discontinuous representation of the surface elvation $\zeta(x, t)$, the superposition integration (8) must be performed in three regions: 1) upstream ($x' \geqq \frac{1}{2}$), 2) under the pressure distribution ($|x'| \leqq \frac{1}{2}$) and 3) downstream ($x' \leqq -\frac{1}{2}$).

The arguments for the exponential integral and exponential terms are given in nondimensional form. First, the wavenumbers are normalized as follows:

$$\gamma_i(\alpha) = K_i/K_0, \quad i = 1, 2, 3, 4.$$

The speed is expressed in terms of the Froude number $F = c/(gl)^{\frac{1}{2}}$, and the normalized longitudinal distance as $x' = x/l$.

For convenience, one may set

$$L_1 = \frac{x' + \frac{1}{2}}{4F^2}, \qquad X = \frac{x'}{4F^2},$$

$$L_2 = \frac{x' - \frac{1}{2}}{4F^2}, \qquad G = \frac{1}{8F^2},$$

$$L_3 = \frac{\frac{1}{2} - x'}{4F^2},$$

$$L_4 = \frac{-(x' + \frac{1}{2})}{4F^2}.$$

The components of the surface elevation are listed as follows:

1. $x' \geqq \frac{1}{2}$ (upstream), $\alpha < 1$

$$a_1 = \frac{1}{2\pi\sqrt{1+\alpha}} \{f(\gamma_1 L_1) - f(\gamma_1 L_2)\},$$

$$a_2 = \frac{1}{2\pi\sqrt{1+\alpha}} \{-f(\gamma_2 L_1) + f(\gamma_2 L_2)\},$$

$$a_3 = \frac{1}{2\pi\sqrt{1-\alpha}} \{f(\gamma_3 L_1) - f(\gamma_3 L_2) + 4\pi \sin(\gamma_3 X) \sin \gamma_3 G\},$$

$$a_4 = \frac{1}{2\pi\sqrt{1-\alpha}} \{-f(\gamma_4 L_1) + f(\gamma_4 L_2)\},$$

$$b_1 = \frac{-1}{2\pi\sqrt{1+\alpha}} \{g(\gamma_1 L_1) - g(\gamma_1 L_2) + \ln(\gamma_1 L_1) - \ln(\gamma_1 L_2)\},$$

$$b_2 = \frac{1}{2\pi\sqrt{1+\alpha}} \{g(\gamma_2 L_1) - g(\gamma_2 L_2) + \ln(\gamma_2 L_1) - \ln(\gamma_2 L_2)\},$$

$$b_3 = \frac{1}{2\pi\sqrt{1-\alpha}} \{g(\gamma_3 L_1) - g(\gamma_3 L_2) + \ln(\gamma_3 L_1) - \ln(\gamma_3 L_2) - 4\pi \cos(\gamma_3 X) \sin(\gamma_3 G)\},$$

$$b_4 = \frac{-1}{2\pi\sqrt{1-\alpha}} \{g(\gamma_4 L_1) - g(\gamma_4 L_2) + \ln(\gamma_4 L_1) - \ln(\gamma_4 L_2)\}.$$

2. $|x'| \leqq \frac{1}{2}$ (under pressure distribution), $\alpha < 1$

$$a_1 = \frac{1}{2\pi\sqrt{1+\alpha}} \{\pi + f(\gamma_1 L_3) + f(\gamma_1 L_1) - 2\pi \cos(\gamma_1 L_3)\},$$

$$a_2 = \frac{-1}{2\pi\sqrt{1+\alpha}} \{\pi + f(\gamma_2 L_3) + f(\gamma_2 L_1) - 2\pi \cos(\gamma_2 L_3)\},$$

$$a_3 = \frac{1}{2\pi\sqrt{1-\alpha}} \{\pi + f(\gamma_3 L_3) + f(\gamma_3 L_1) + -2\pi \cos(\gamma_3 L_1)\},$$

$$a_4 = \frac{-1}{2\pi\sqrt{1-\alpha}}\{\pi + f(\gamma_4 L_3) - f(\gamma_4 L_1) - 2\pi\cos(\gamma_4 L_3)\},$$

$$b_1 = \frac{-1}{2\pi\sqrt{1+\alpha}}\{g(\gamma_1 L_1) - g(\gamma_1 L_3) + \ln(\gamma_1 L_1) - \ln(\gamma_1 L_3) + 2\pi\sin(\gamma_1 L_3)\},$$

$$b_2 = \frac{1}{2\pi\sqrt{1+\alpha}}\{g(\gamma_2 L_1) - g(\gamma_2 L_3) + \ln(\gamma_2 L_1) - \ln(\gamma_2 L_2) + 2\pi\sin(\gamma_2 L_3)\},$$

$$b_3 = \frac{-1}{2\pi\sqrt{1-\alpha}}\{g(\gamma_3 L_3) - g(\gamma_3 L_1) + \ln(\gamma_3 L_3) - \ln(\gamma_3 L_1) + 2\pi\sin(\gamma_3 L_1)\},$$

$$b_4 = \frac{1}{2\pi\sqrt{1-\alpha}}\{g(\gamma_4 L_3) - g(\gamma_4 L_1) + \ln(\gamma_4 L_3) - \ln(\gamma_4 L_4) - 2\pi\sin(\gamma_4 L_4)\}.$$

3. $x' \leqq -\frac{1}{2}$ (downstream), $\alpha < 1$

$$a_1 = \frac{-1}{2\pi\sqrt{1+\alpha}}\{f(\gamma_1 L_4) - f(\gamma_1 L_3) + 4\pi\sin(\gamma_1 X)\sin(\gamma_1 G)\},$$

$$a_2 = \frac{1}{2\pi\sqrt{1+\alpha}}\{f(\gamma_2 L_4) - f(\gamma_2 L_3) + 4\pi\sin(\gamma_2 X)\sin(\gamma_2 G)\},$$

$$a_3 = \frac{-1}{2\pi\sqrt{1-\alpha}}\{f(\gamma_3 L_4) - f(\gamma_4 L_3)\},$$

$$a_4 = \frac{1}{2\pi\sqrt{1-\alpha}}\{f(\gamma_4 L_4) - f(\gamma_4 L_3) + 4\pi\sin(\gamma_4 X)\sin(\gamma_4 G)\},$$

$$b_1 = \frac{-1}{2\pi\sqrt{1+\alpha}}[g(\gamma_1 L_4) - g(\gamma_1 L_3) + \ln(\gamma_1 L_4) - \ln(\gamma_1 L_3)$$
$$+ 4\pi\cos(\gamma_1 X)\sin(\gamma_1 G)],$$

$$b_2 = \frac{1}{2\pi\sqrt{1+\alpha}}[g(\gamma_2 L_4) - g(\gamma_2 L_3) + \ln(\gamma_2 L_4) - \ln(\gamma_2 L_3)$$
$$+ 4\pi\cos(\gamma_2 X)\sin(\gamma_2 G)],$$

$$b_3 = \frac{1}{2\pi\sqrt{1-\alpha}}[g(\gamma_3 L_4) - g(\gamma_3 L_3) + \ln(\gamma_3 L_4) - \ln(\gamma_3 L_3)],$$

$$b_4 = \frac{-1}{2\pi\sqrt{1-\alpha}}[g(\gamma_4 L_4) - g(\gamma_4 L_3) + \ln(\gamma_3 L_4) - \ln(\gamma_4 L_3)$$
$$+ 4\pi\cos(\gamma_4 X)\sin(\gamma_4 G)].$$